Game Theory

- Normative approach
- *Interactive, separate* decision making: the players make their decisions independently of each other (there is *no collusion*)
- Separate choices interact to determine a pair of payoffs for each side

Types of games

- *Non-cooperative* games
- *Cooperative* games
- *Simultaneous* games
- *Sequential* games
- Games with *complete* information
- Games with *incomplete* information

GT's Assumptions

- You have to *act* (Doing nothing is an act.)
- Your *payoff depends* both on what you do and on what other designated players do.
- You do not know what they ***will*** do – but you know what they ***could*** do.
- They do not what you will do.
- Players are *rationale*.

The Rules of the Game

- *Fixed strategies* (you have to choose one)
- *Two alternatives*
- *Perfect information* (knowledge of all possible consequences and of each other's preferences)
- *Common knowledge* (of the choice sets)
- *Simultaneous choices*
- *No cheap talk* (no preplay discussion)

NASH Equilibrium

- Two strategies are in ***Nash equilibrium*** if each agent's strategy is a best response to its opponent's strategy.
- Given that one agent is using this strategy, it is not beneficial for the other agent to use another different strategy.
- This is a necessary condition for system *stability* where both agents act strategically.

The Matrix display

COLUMN PLAY

ROW PLAYER		C1 (LEFT)	C
	R1 (UP)	R1,C1	R1,C2
	R2 (DOWN)	R2,C1	R2

The aim is not to get more than the other side, but to get as much for your side as possible

Each player assigns a utility value to each of the four consequences (*common knowledge*)

Game 1: Indeterminacy

		COL	
		LEFT	RIGHT
ROW	UP	(4,5)	(10,-6)
	DOWN	(12,7)	(5,9)

No simple fixed strategy for either player is **clearly better** than any other

Game 2: Dominance

ROW \ COL	LEFT	RIGHT
UP	(4,3)	(3,0)
DOWN	(12,8)	(5,4)

X ***dominates*** strategy Y if you are always better off with X

Game 3: Iterative Dominance

COL

Left is a non contender

		LEFT	RIGHT
ROW	UP	(0,2)	(5,4)
	DOWN	(10,3)	(3,8)

Not clear what to do

The pair is in **equilibrium**: no motivation for either player to change if the other holds fixed

Game 5

COL

			LEFT	RIGHT
p	*ROW*	UP	(4,-100)	(10,6)*
1-p		DOWN	(12,8)*	(5,4)

Col should take Right if he thinks that the *chance of Row's choosing UP* is larger than 0.036

-100p+8(1-p) for *Left*

6p+ 4(1-p) for *Right*

The *break-eve* probability:

-100p+8(1-p)= 6p+ 4(1-p)

p=0.036

Game 6: The Battle of Sexes

HUSBAND

WIFE	CINEMA	FOOTBALL
FOOTBALL	(0,0)	(1,2)*
CINEMA	(2,1)*	(0,0)

Each player prefers one of the equilibria. If one player can choose first…

Game 7: Threats and Communication

COL

ROW		LEFT	RIGHT
	UP	(1,5)	(4,1)
	DOWN	(0,-10)	(1,-20)

Sometimes you may be better off not negotiating.

If Colin comes to negotiating table he may be forced by Rowena to choose Right

The Prisoner's Dilemma (A. Tucker, 1953)

- Confessing or not confessing
- Prisoners are kept separated
- If *neither confesses*: one year for possession of an illegal weapon
- If they *both confess*: three years for both
- If *one confesses* and the *other doesn't*: the confessor will get off scot-free and the non confessor will be given five years.
- That's the social trap

Game 8: the Dilemma Game

	NOT CONFESS	CONFESS
NOT CONFESS	(1,1)	(5,0)
CONFESS	(0,5)	(3,3)

Uncoordinated, rational, self-interested agent can result in awful outcome

Game 9: the game of “Chicken”

	CHICKEN	MACHO
CHICKEN	I*	III*
MACHO	II*	[illegible]

“I” is not an equilibrium of the <u>one-round game</u>, because each player would prefer to be macho if he expected the other player be cautious

Game 10: the iterated Dilemma Game

- Play game repeatedly
- No talking
- C= Cooperate
- D= Defect
- The game is symmetric (row player)

Game 10: the iterated Dilemma Game

	Cooperate	Defect
Cooperate	(5,5)	(-5,10)
Defect	(10,-5)	(-2,-2)

	1	2	3	4	5	6	7	8	9	10
ROW	C	C	C	C	C	C	C	C	…	…
COL	C	C	C	C	C	C	C	C	…	…

Game 11: the ultimatum game (Thaler)

- Prize to be shared: $100
- Players: Allocator (A) Recipient (R)
- Move 1: A offers allocation of X to R and (100-X) to A
- Move 2: R accepts X and gives (100-X) to A – or R rejects allocation and both get nothing
- X= the ultimatum bid
- Rules of play: R has veto power, no preplay communication, single-shot game

↓

Game with many perfect equilibrium pairs

Nash equilibrium

- DEF. In a game with n players $G=\{S_1, \ldots, S_n; u_1,\ldots, u_n\}$ the strategies $(s_1^*, \ldots, s_n^*)$ are in Nash equilibrium if, for each player i, s_i^* is the better response of player i to the strategy specified for the other $n-1$ players $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$:

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*)$$

for all possible strategies s_i in S_i

The Battle of Sexes (1)

		HUSBAND	
		CINEMA	FOOTBALL
WIFE	FOOTBALL	(0,0)	(1,3)*
	CINEMA	(3,1)*	(0,0)

The Battle of Sexes (2)

- Seemingly, you have only two choices: Football(a) or Cinema (b)
- Actually, there are not the only choices you have:
  - You can toss a coin (c)!
- You have one more strategy, but the same set of actions!
- a, b *pure strategies*
- *c mixed strategy* (just one of many)

- **Definition**: Suppose a player has M pure strategies, s^1, s^2, s^3,…, s^M. A mixed strategy for this player is a *probability distribution* over his pure strategies; that is, it is a probability vector $(p^1, p^2,\ldots, p^M)$, with $p^k \geq 0$, k=1,…,M, and $\sum_{K=1}^{M} p^k = 1$

Observations

- Each pure strategy is – in a trivial sense – also a mixed strategy
- A mixed strategy can be evaluated through the Von-Neumann-Morgenstern utility function *(espected payoff).*

- Es. The Battle of Sexes

An implication

- **Def.**Consider a mixed strategy given by the probability vector $(p^1, p^2,\ldots, p^M)$. The <u>support</u> of this mixed strategy is given by all those pure strategies that have a positive probability of getting played.
  - Es. $p^1>0$ and $p^3>0$, but $p^k=0$ for all other k, then the support is made up of the pure strategies 1 and 3.
- **Implication.** (a) A mixed strategy $(p^1, p^2,\ldots, p^M)$ is a *best response* to s_{-i} if and only if each of the pure strategies in its support is itself a best response to s_{-i}. (b) In that case *any* mixed strategy over this support will be a best response.

No-Name game

Mixed strategy rationale

1. Mixed strategies may dominate pure strategies (that are themselves undominated by other pure strategies).
2. The worst-case payoff of a mixed strategy may be better than the worst-case payoff of every pure strategy
3. If we restrict ourselves to pure strategies, we may not be able to find Nash equilibrium to a game.

Existence of Nash equilibrium

- **Theorem (Nash, 1950):** Suppose that a game has a finite number of strategies for each player. Then there is at least one Nash equilibrium (possible in mixed strategies).

Compare three lotteries

- Expected utility assignment to lottery A – tickets for Champions League's final that could cost 20, 25, 30, 35 dollars with equal probabilities:

$$1/4u(20,G)+ 1/4u(25,G)+ 1/4u(30,G)+ 1/4u(35,G)$$

- Expected utility assignment to lottery B – stay in and study for the game theory exam:

$$0.5u(5,S)+ 0.4u(10,S)+0.1u(15,S)$$

- Expected utility assignment to lottery C – go to the pub:

$$u(10, P)$$

Expected Utility Theory

- A *simple lottery L* is a list $L=(p_1, \ldots, p_n)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$ where p_n is interpreted as the probability of outcome *n* occuring.

- We take the set of alternatives, denoted here by $\mathcal{L}$, to be the *set of all simple lotteries over the set of outcomes* C.
- The decision maker has a rational preference relation $\succeq$ on $\mathcal{L}$ a complete and transitive relation allowing comparison of any pair of simple lotteries.

- **Def.** The utility function $U: \mathscr{L} \rightarrow \Re$ has an expected utility form if there is an assignment of numbers $(u_1, \ldots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, \ldots, p_N) \in \mathscr{L}$ we have:

$$U(L) = u_1 p_1 + \ldots + u_N p_N.$$

P.S. the utility of a lottery can be thought of as expected value of the utilities u_n of the N outcomes.

An example of utility function

- The outcomes are monetary (dollars)
- The utility function over outcomes is:

$$u(x)= \log(10+x)$$

- What is the expected utility of the lottery: "Win $5 if heads, lose $5 if tails"?

$$0{,}5\log 15 + 0{,}5\log 5$$

An example of utility function (2)

- Win 25$ with probability 0.3, win $18 with probability 0.1, lose $8 with probability 0.4, and lose $2 with probability 0.2" has expected utility equal to:

0.3 log 35+ 0.1 log 28+ 0.4 log 2+ 0.2 log 8

Von Neumann-Morgenstern Restrictions

- Monotonicity
- Archimedean
- Substitution
- Compound lotteries are equivalent to a simple lotteries with the same distribution over final outcomes

Expected Utility Theorem II

- Under these restrictions there is a utility representation *u* over sure outcomes such that lottery A is at least as good as lottery B if and only if the expected utility to lottery A is at least as high as the expected utility to lottery B.

The Extensive Form Games

- Game tree:
  - Root
  - Branches
  - Decision node
  - Terminal node (payoff)

- Es. Theater game

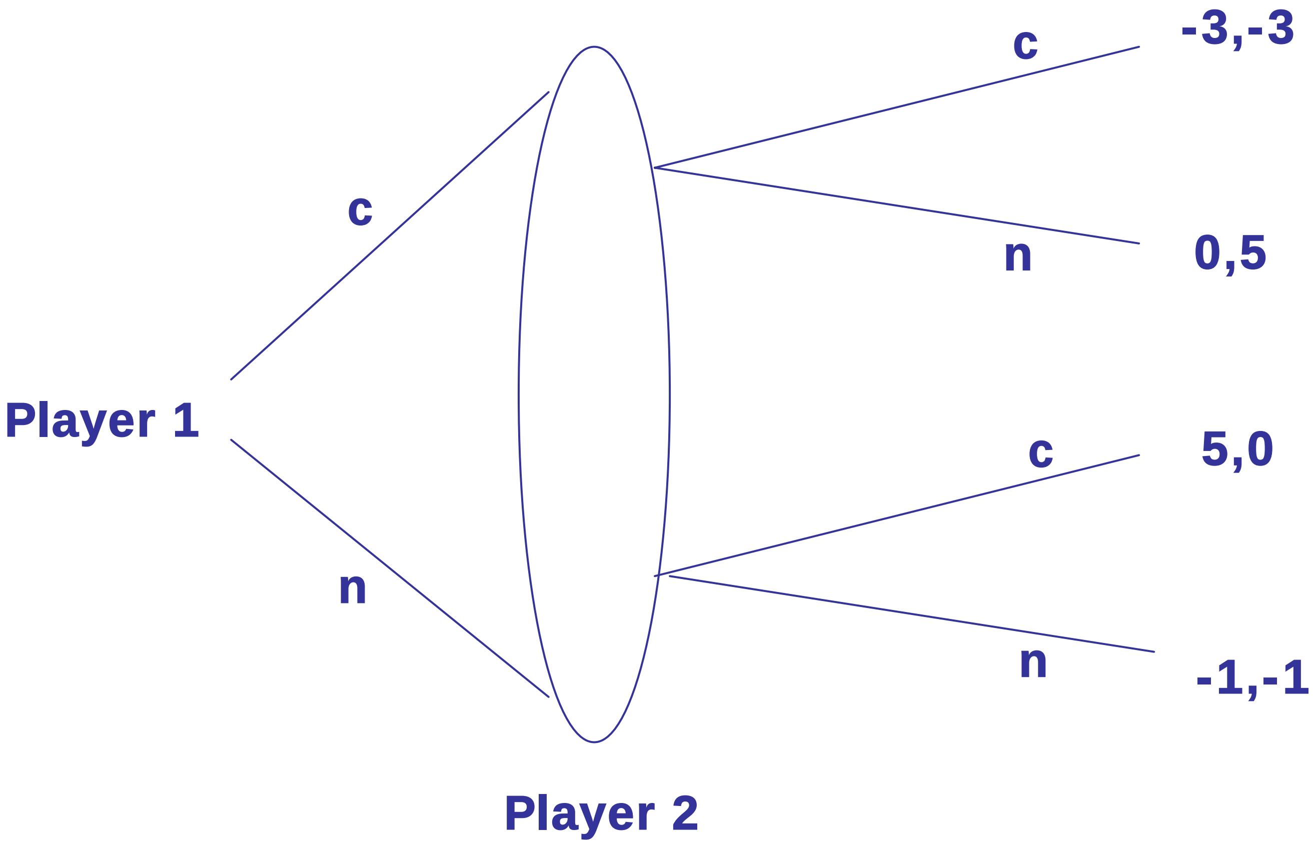
Requirements

- Single starting point
- No cycles
- One way to proceed

Game 8: the Dilemma Game

	NOT CONFESS	CONFESS
NOT CONFESS	(-1,-1)	(5,0)
CONFESS	(0,5)	(-3,-3)

Extensive form



Perfect information game

- A game of **perfect information** is one in which there is no information set (with multiple nodes).
- Es. Coke vs. Pepsi

Bibliography

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